Question 1 — Integral Calculus **(8 marks)**

Water flows into a tank at a rate modelled by $V'(t) = 6t - t^2$ (litres per minute), for $0 \le t \le 6$, where V(t) is the volume of water in the tank, in litres, after t minutes. Initially, the tank contains 10 litres of water.

(a) Determine V(t). (3 marks)

Solution

$$V(t) = \int (6t - t^2)\, dt = 3t^2 - \frac{t^3}{3} + C$$

$$V(0) = 10 \implies C = 10$$

$$V(t) = 3t^2 - \frac{t^3}{3} + 10$$

Specific behaviours

✓ correct antiderivative, $3t^2 - t^3/3 + C$

✓ correctly uses the initial condition $V(0) = 10$ to find $C = 10$

✓ correct final expression, $V(t) = 3t^2 - t^3/3 + 10$

(b) Determine the volume of water in the tank after 3 minutes. (2 marks)

Solution

$$V(3) = 3(3)^2 - \frac{3^3}{3} + 10$$

$$V(3) = 27 - 9 + 10 = 28 \text{ litres}$$

Specific behaviours

✓ correct substitution of $t = 3$ into $V(t)$

✓ correct value, $V(3) = 28$ litres

(c) Evaluate the definite integral $\int (6t - t^2)\, dt$ from $t = 0$ to $t = 6$, and interpret this value in the context of the tank. (3 marks)

Solution

$$\int (6t - t^2)\, dt \text{ from 0 to 6} = \left(3(6)^2 - \frac{6^3}{3}\right) - \left(3(0)^2 - \frac{0^3}{3}\right)$$

$$= (108 - 72) - 0 = 36$$

This represents the total change in the volume of water in the tank between $t = 0$ and $t = 6$ minutes — the volume increases by 36 litres over this interval.

Specific behaviours

✓ correctly evaluates the antiderivative at the upper and lower limits

✓ correct value of the definite integral, 36

✓ correct interpretation: the volume of water increases by 36 litres over the 6 minutes

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Question 2 — Discrete Random Variables **(7 marks)**

A discrete random variable X has probability distribution given by $P(X = x) = kx$ for $x = 1, 2, 3, 4$, and $P(X = x) = 0$ otherwise, where k is a positive constant.

(a) Show that $k = 0.1$. (2 marks)

Solution
$k + 2k + 3k + 4k = 1$, since the probabilities must sum to 1
$10k = 1 \Longrightarrow k = 0.1$

Specific behaviours
✓ uses the condition that the probabilities sum to 1 to form $10k = 1$
✓ correct value, $k = 0.1$

(b) Determine E(X). (2 marks)

Solution
$P(X = 1) = 0.1$, $P(X = 2) = 0.2$, $P(X = 3) = 0.3$, $P(X = 4) = 0.4$
$E(X) = 1(0.1) + 2(0.2) + 3(0.3) + 4(0.4) = 0.1 + 0.4 + 0.9 + 1.6$
$E(X) = 3$

Specific behaviours
✓ correct computation of $\Sigma x \cdot P(x)$ using the probabilities from (a)
✓ correct value, $E(X) = 3$

(c) Determine Var(X). (3 marks)

Solution
$E(X^2) = 1^2(0.1) + 2^2(0.2) + 3^2(0.3) + 4^2(0.4) = 0.1 + 0.8 + 2.7 + 6.4$
$E(X^2) = 10$
$\text{Var}(X) = E(X^2) - [E(X)]^2 = 10 - 3^2 = 1$

Specific behaviours
✓ correctly finds $E(X^2) = 10$
✓ correct use of $\text{Var}(X) = E(X^2) - [E(X)]^2$
✓ correct value, $\text{Var}(X) = 1$

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Question 3 — The Normal Distribution **(8 marks)**

The diameters of ball bearings produced by a machine are normally distributed with a mean of 50 mm and a standard deviation of 2 mm. Let X be the diameter, in mm, of a randomly selected ball bearing produced by the machine. You may use the fact that approximately 68% of a normal distribution lies within 1 standard deviation of the mean, 95% lies within 2 standard deviations, and 99.7% lies within 3 standard deviations.

(a) Determine the z-score corresponding to a diameter of 54 mm. (2 marks)

Solution

$$z = \frac{54 - 50}{2}$$

$$z = 2$$

Specific behaviours

✓ correct use of $z = (x - \mu)/\sigma$

✓ correct value, $z = 2$

(b) Determine $P(46 < X < 54)$. (2 marks)

Solution

46 and 54 are each 2 standard deviations from the mean (50 ± 4).

$$P(46 < X < 54) \approx 0.95$$

Specific behaviours

✓ identifies that 46 and 54 lie 2 standard deviations either side of the mean

✓ correct probability, 0.95

(c) Determine $P(X > 52)$. (3 marks)

Solution

52 is 1 standard deviation above the mean (50 + 2).

$$\text{By symmetry, } P(X > 52) = \frac{1 - 0.68}{2}$$

$$P(X > 52) = 0.16$$

Specific behaviours

✓ identifies that 52 is 1 standard deviation above the mean

✓ correct use of symmetry to find the upper-tail probability, $(1 - 0.68)/2$

✓ correct value, $P(X > 52) = 0.16$

(d) The machine rejects any ball bearing with a diameter more than 3 standard deviations from the mean. State the range of diameters that will be accepted. (1 mark)

Solution

3 standard deviations = 6 mm

Accepted range: 50 − 6 to 50 + 6, i.e. 44 mm to 56 mm

Specific behaviours

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✓ correct range, 44 mm ≤ diameter ≤ 56 mm

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